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Reflection and refraction at a face of a cholesteric liquid crystal

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Abstract. The theory of electromagnetic wave propagation in a cholesteric liquid crystal is developed, in the case when the free-space wavelength of the radiation is small compared to the pitch of the crystal. An arbitrary direction of the light is considered. Reflection and refraction at a crystal face that is perpendicular to the cholesteric helix is studied. Explicit formulae are found for the electric and magnetic fields of the reflected and refracted beams.

1. Introduction

Consider a cholesteric liquid crystal with a face perpendicular to the helical axis. Suppose light of some definite frequency and polarization is incident on the crystal at some arbitrary angle. The problem considered here is to find the properties of the light reflected from the crystal and refracted into the crystal.

There are very well known formulae for light propagation parallel to the cholesteric axis. They were found originally by Mauguin [1], Oseen [2] and de Vries [3]; de Gennes [4] has reviewed the development. For non-axial propagation the problem is considerably more complicated. Analytically the problem is formidable, so many of the early treatments were entirely numerical. However, Peterson [5] and Oldano *et al* [6] gave analytic solutions for the propagation of the waves in the crystal. The results are somewhat complicated, so applications are made numerically. Thus the implications of the theory can be found in any specific case. However, brief formulae and concise physical pictures of the reflection and refraction phenomena do not, in general, exist.

The purpose of this paper is to show that the problem can be solved analytically in all detail in the case when the light wavelength is small compared to the pitch of the cholesteric helix, what de Gennes calls the Mauguin limit. In contrast to the general case, there are brief formulae for all quantities and a concise interpretation of the results. It is felt that this limit is of interest in itself and also will be valuable as a special case in future treatments of the general problem.

The problem is solved by applying the process known as the geometric optics method or the WKB (Wentzel–Kramers–Brillouin) method to Maxwell's equations. In liquid-crystal applications this method has been applied formerly by Ong and Meyer [7] to a periodically bent nematic crystal and by Good [8] for axial propagation in a crystal with arbitrary variation of the director from plane to plane.

The transmitted wave propagates in the plane of the incident and reflected waves. It is found that there is an ordinary and an extraordinary wave inside. The ordinary wave has a simple space dependence with a definite wavevector. The extraordinary wave has a more complicated space dependence. It does not have a definite wavevector but, periodically in space, it coincides with a wave that does have a definite vector. Thus one can keep track of the actual wave by following the coinciding wave, which has a well defined phase and group velocity. It is convenient to separate the reflection–refraction process into a component where there is only an ordinary transmitted wave and a component where there is only an extraordinary transmitted wave. Formulae for all components of all the waves, reflected and transmitted, are found. All the waves are plane-polarized. There is an analogue of the Brewster's angle effect; at two special incident directions and polarizations relative to the director on the face, there is a transmitted wave but no reflected wave. The method suggests when there will be no transmitted wave, due to the Bragg reflection effect, but it does not give information on the width of the stop bands.

2. Basic equations

Let the coordinate axes be chosen so the crystal is in the region $z > 0$. The z axis is the cholesteric axis and the reflection and refraction takes place at the $z = 0$ plane. Also let the direction of the incident radiation be in the $y = 0$ plane, having positive x and z components.

Maxwell's equations, in SI units, are

$$\nabla \times \boldsymbol{H} = \partial \boldsymbol{D}/\partial t \qquad \nabla \times \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t$$
$$\nabla \cdot \boldsymbol{B} = 0 \qquad \nabla \cdot \boldsymbol{D} = 0. \tag{1}$$

As usual, complex solutions will be found and the real parts may be taken at the end of the calculation. The time dependence $\exp(-\mathrm{i}\omega t)$ is assumed, so the divergence equations are satisfied automatically. One uses $\boldsymbol{B} = \mu_0 \boldsymbol{H}$, as a property of the material, to eliminate the magnetic field and reduce the problem to

$$\nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E} = \mu_0 \omega^2 \boldsymbol{D}. \tag{2}$$

For a cholesteric liquid crystal there is a tensor dielectric constant such that

$$D_\alpha = \varepsilon_0 \varepsilon_{\alpha\beta} E_\beta \tag{3}$$

where

$$\varepsilon_{\alpha\beta} = \varepsilon_\perp \delta_{\alpha\beta} + (\varepsilon_\parallel - \varepsilon_\perp) n_\alpha n_\beta. \tag{4}$$

Here $\varepsilon_\parallel$ and $\varepsilon_\perp$ are constants but the director $\boldsymbol{n}$ varies with z according to

$$n_x = \cos\Phi \qquad n_y = \sin\Phi \qquad n_z = 0 \tag{5}$$

where

$$\Phi = qz + \Phi_0. \tag{6}$$

Here q may be positive or negative, depending on whether the helical-type dependence is right-handed or left-handed; the pitch of the helix is $2\pi/|q|$; the system repeats in a

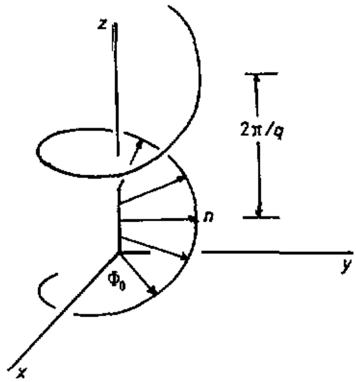

Figure 1. The variation of the director $\boldsymbol{n}$ with z as given by (5) and (6), in the case of a right-handed helical dependence. The director is always parallel to the xy plane, making angle Φ with the x axis. On the crystal face, at $z = 0$, the angle is Φ_0. For the purpose of illustration, the director is shown as originating at the z axis for various values of z. However $\boldsymbol{n}(z)$ applies for all the molecules in a plane at a specific value of z.

distance $\pi/|q|$. The director is at angle Φ_0 to the x axis on the face of the crystal. The variation of $\boldsymbol{n}$ with z is illustrated in figure 1. In detail the components of $\varepsilon_{\alpha\beta}$ are

$$
\begin{aligned}
\varepsilon_{xx} &= \tfrac{1}{2}(\varepsilon_{\|} + \varepsilon_{\perp}) + \tfrac{1}{2}(\varepsilon_{\|} - \varepsilon_{\perp})\cos(2\Phi) \\
\varepsilon_{yy} &= \tfrac{1}{2}(\varepsilon_{\|} + \varepsilon_{\perp}) - \tfrac{1}{2}(\varepsilon_{\|} - \varepsilon_{\perp})\cos(2\Phi) \\
\varepsilon_{zz} &= \varepsilon_{\perp} \\
\varepsilon_{xy} &= \varepsilon_{yx} = \tfrac{1}{2}(\varepsilon_{\|} - \varepsilon_{\perp})\sin(2\Phi) \\
\varepsilon_{xz} &= \varepsilon_{zx} = \varepsilon_{yz} = \varepsilon_{zy} = 0.
\end{aligned}
\tag{7}
$$

Let the x dependence of the fields be $\exp(ilx)$ throughout; there is no y dependence. The components of (2) can be written as

$$
\mathrm{i}l\,\partial E_z/\partial z - \partial^2 E_x/\partial z^2 = \tfrac{1}{2}(\omega/c)^2\{(\varepsilon_{\|} + \varepsilon_{\perp})E_x + (\varepsilon_{\|} - \varepsilon_{\perp})[\cos(2\Phi)E_x + \sin(2\Phi)E_y]\}
\tag{8}
$$

$$
l^2 E_y - \partial^2 E_y/\partial z^2 = \tfrac{1}{2}(\omega/c)^2\{(\varepsilon_{\|} + \varepsilon_{\perp})E_y + (\varepsilon_{\|} - \varepsilon_{\perp})[\sin(2\Phi)E_x - \cos(2\Phi)E_y]\}
\tag{9}
$$

$$
\mathrm{i}l\,\partial E_x/\partial z + l^2 E_z = (\omega/c)^2 \varepsilon_{\perp} E_z.
\tag{10}
$$

These equations suggest that the abbreviations

$$
\begin{aligned}
k_1^2 &= \tfrac{1}{2}(\omega/c)^2(\varepsilon_{\|} - \varepsilon_{\perp}) \\
k_2^2 &= (\omega/c)^2\varepsilon_{\perp} \qquad \alpha = l^2/k_2^2
\end{aligned}
\tag{11}
$$

be used. Then (10) is solved for E_z as

$$
E_z = [\mathrm{i}l/(k_2^2 - l^2)](\partial E_x/\partial z)
\tag{12}
$$

and E_x and E_y are to be found from the coupled set

$$
\begin{aligned}
&[1/(1-\alpha)](\partial^2 E_x/\partial z^2) + (k_1^2 + k_2^2)E_x + k_1^2[\cos(2\Phi)E_x + \sin(2\Phi)E_y] = 0 \\
&\partial^2 E_y/\partial z^2 + (k_1^2 + k_2^2 - \alpha k_2^2)E_y + k_1^2[\sin(2\Phi)E_x - \cos(2\Phi)E_y] = 0.
\end{aligned}
\tag{13}
$$

3. Approximate solutions

For the WKB approximation one makes the substitutions

$$E_x = (E_x^{(0)} + hE_x^{(1)} + \ldots)\exp\left(\frac{i}{h}\int p\,dz\right)$$
$$E_y = (E_y^{(0)} + hE_y^{(1)} + \ldots)\exp\left(\frac{i}{h}\int p\,dz\right) \tag{14}$$

into (13) and develops a series solution in powers of h. Here h is an index that identifies the orders of the solution and then is set equal to unity. The approximation is made by truncating the series, and the validity of the process is considered after the fact, as discussed in the earlier paper [8]. In the application to (13), l and ω, and hence k_1 and k_2, are considered of order h^{-1} since lx and ωt contribute to the phase of the waves like $\int p\,dz/h$. This is closely related to the geometrical optics approximation, where the expansion is made on ω.

The first contribution, coming from terms proportional to h^{-2}, is to be found from

$$[-p^2/(1-\alpha) + k_1^2 + k_2^2 + k_1^2\cos(2\Phi)]E_x^{(0)} + k_1^2\sin(2\Phi)E_y^{(0)} = 0$$
$$k_1^2\sin(2\Phi)E_x^{(0)} + [-p^2 + k_1^2 + k_2^2 - \alpha k_2^2 - k_1^2\cos(2\Phi)]E_y^{(0)} = 0. \tag{15}$$

The next order terms, proportional to h^{-1}, give the equations

$$(-p^2/(1-\alpha) + k_1^2 + k_2^2 + k_1^2\cos(2\Phi))E_x^{(1)} + k_1^2\sin(2\Phi)E_y^{(1)}$$
$$= -[1/(1-\alpha)](i\,(dp/dz)E_x^{(0)} + 2ip(dE_x^{(0)}/dz)) \tag{16}$$
$$k_1^2\sin(2\Phi)E_x^{(1)} + [-p^2 + k_1^2 + k_2^2 - \alpha k_2^2 - k_1^2\cos(2\Phi)]\,E_y^{(1)}$$
$$= -(i\,(dp/dz)E_y^{(0)} + 2ip\,(dE_y^{(0)}/dz)).$$

Equations (15) have a solution for $E_x^{(0)}, E_y^{(0)}$ only if the determinant of the coefficients is zero. This condition yields two possible values of p^2:

$$p_{or}^2 = (1-\alpha)k_2^2 \tag{17}$$
$$p_{ex}^2 = (1-\alpha)(2k_1^2 + k_2^2) + 2k_1^2\alpha\sin^2\Phi. \tag{18}$$

In terms of ω and l these values are

$$p_{or}^2 = \varepsilon_\perp(\omega/c)^2 - l^2$$
$$p_{ex}^2 = \varepsilon_\parallel(\omega/c)^2 - (l^2/\varepsilon_\perp)(\varepsilon_\perp\sin^2\Phi + \varepsilon_\parallel\cos^2\Phi).$$

The wave that satisfies (17) is called ordinary because p_{or} is constant, the phase of the wave depends on $(lx + p_{or}z - \omega t)$, and (17) gives the dispersion equation

$$\omega^2 = (c^2/\varepsilon_\perp)(p_{or}^2 + l^2). \tag{19}$$

The wave that satisfies (18) is extraordinary. For each of the allowed values of p^2, one can solve (15); let

$$\begin{pmatrix} E_x^{(0)} \\ E_y^{(0)} \end{pmatrix} = C\begin{pmatrix} m_x \\ m_y \end{pmatrix} \tag{20}$$

where m is normalized as discussed below and C is chosen so as to allow solution of the next order. That is, (16) is solvable for $E_x^{(1)}$, $E_y^{(1)}$ only if the solution of the corresponding homogeneous set of equations, already known to be Cm, is orthogonal to the vector on the right. This condition leads to the equation

$$[m_x^2/(1-\alpha)+m_y^2][(\mathrm{d}p/\mathrm{d}z)C+2p(\mathrm{d}C/\mathrm{d}z)]+pC(\mathrm{d}/\mathrm{d}z)[m_x^2/(1-\alpha)+m_y^2]=0.$$

Evidently the appropriate normalization is

$$m_x^2/(1-\alpha)+m_y^2=1 \tag{21}$$

and the necessary value of C is

$$C=\bar{C}p^{-1/2} \tag{22}$$

where $\bar{C}$ is a constant. The solutions of (15) with this normalization are

$$\begin{pmatrix} m_x \\ m_y \end{pmatrix}_{\text{or}} = \left(\frac{1-\alpha}{1-\alpha\cos^2\Phi}\right)^{1/2}\begin{pmatrix} -\sin\Phi \\ \cos\Phi \end{pmatrix}$$

$$\begin{pmatrix} m_x \\ m_y \end{pmatrix}_{\text{ex}} = \left(\frac{1}{1-\alpha\cos^2\Phi}\right)^{1/2}\begin{pmatrix} (1-\alpha)\cos\Phi \\ \sin\Phi \end{pmatrix}. \tag{23}$$

The first approximation is found by keeping the $E^{(0)}$ terms only. The results for the x, y components of the electric fields of the waves propagating into the material are

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix}_{\text{or}} = \frac{\bar{C}_{\text{or}}}{(1-\alpha\cos^2\Phi)^{1/2}}\begin{pmatrix} -\sin\Phi \\ \cos\Phi \end{pmatrix}\exp[\mathrm{i}(lx+p_{\text{or}}z-\omega t)] \tag{24}$$

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix}_{\text{ex}} = \frac{\bar{C}_{\text{ex}}}{(p_{\text{ex}})^{1/2}(1-\alpha\cos^2\Phi)^{1/2}}\begin{pmatrix} (1-\alpha)\cos\Phi \\ \sin\Phi \end{pmatrix}\exp\left[\mathrm{i}\left(lx+\int_0^z p_{\text{ex}}\,\mathrm{d}z-\omega t\right)\right]. \tag{25}$$

Here the integration limits have been chosen so the boundary conditions at $z = 0$ can be easily applied. The z components of the fields are to be found from (12) where, consistently taking the first approximation, one replaces $\partial/\partial z$ by $\mathrm{i}p$. Thus

$$E_z=-[lp/(k_2^2-l^2)]E_x \tag{26}$$

in each mode. Similarly $\boldsymbol{B}$ for each mode is found from $(-\mathrm{i}/\omega)\nabla\times\boldsymbol{E}$ to be

$$\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \frac{1}{\omega}\begin{pmatrix} -pE_y \\ [p/(1-\alpha)]E_x \\ lE_y \end{pmatrix}. \tag{27}$$

It is interesting that the electric field of the ordinary wave is always perpendicular to the director, and the magnetic field of the extraordinary wave is always perpendicular to the director.

The behaviour of the extraordinary wave is complicated by the $\int p_{ex}\,dz$ term in the phase. However, the periodicity of the integrand suggests a further discussion of this wave. Consider the planes where

$$|q|z = 2\pi n \qquad n = 0, 1, 2, \ldots. \tag{28}$$

On these planes $\cos\Phi$ and $\sin\Phi$ always have the same values, $\cos\Phi_0$ and $\sin\Phi_0$. Also the integral between every two adjacent planes has the value

$$\int_0^{2\pi/|q|} p_{ex}\,dz = (4\omega/|q|c)(\varepsilon_\parallel - \alpha\varepsilon_\perp)^{1/2}E(m). \tag{29}$$

Here (18) was used for p_{ex} and the integral was expressed in terms of the complete elliptic integral of the second kind,

$$E(m) = \int_0^{\pi/2} (1 - m\sin^2\theta)^{1/2}\,d\theta \tag{30}$$

with the parameter

$$m = (\varepsilon_\parallel - \varepsilon_\perp)\alpha/(\varepsilon_\parallel - \alpha\varepsilon_\perp). \tag{31}$$

Abramowitz and Stegun [9] give a discussion and tables of the complete elliptic integrals $E(m)$ and $K(m)$ used below. On the nth plane the value of the field is

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix}_{ex} = \frac{\bar{C}_{ex}}{[p_{ex}(0)]^{1/2}[1 - \alpha\cos^2\Phi_0]^{1/2}} \begin{pmatrix} (1-\alpha)\cos\Phi_0 \\ \sin\Phi_0 \end{pmatrix} \times \exp\{i[lx + (4\omega/|q|c)(\varepsilon_\parallel - \alpha\varepsilon_\perp)^{1/2}E(m)n - \omega t]\}$$

where $p_{ex}(0)$ is p_{ex} at $z = 0$. Consequently one considers the field

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix}_{co} = \frac{\bar{C}_{ex}}{(p_{ex})^{1/2}(1 - \alpha\cos^2\Phi)^{1/2}} \begin{pmatrix} (1-\alpha)\cos\Phi \\ \sin\Phi \end{pmatrix} \exp[i(lx + rz - \omega t)] \tag{32}$$

where

$$r = (2\omega/\pi c)(\varepsilon_\parallel - \alpha\varepsilon_\perp)^{1/2}E(m). \tag{33}$$

The point is that this field has a definite wavenumber $(l, 0, r)$ and it coincides with the extraordinary field on every one of the planes $z = 2\pi n/|q|$. Thus one can keep track of the extraordinary wave by following this coinciding wave. Equation (33) gives implicitly ω as a function of l and r, so gives the dispersion of the coinciding wave. One can take partial derivatives through (33) to get the group velocity of the coinciding wave. The results are

$$\begin{aligned} V_x &= (\partial\omega/\partial l)|_r = \{[\varepsilon_\parallel(K - E) + \alpha\varepsilon_\perp E]/(\alpha\varepsilon_\perp)^{1/2}\varepsilon_\parallel K\}c \\ V_z &= (\partial\omega/\partial r)|_l = \{(\pi/2)(\varepsilon_\parallel - \alpha\varepsilon_\perp)^{1/2}/\varepsilon_\parallel K\}c \end{aligned} \tag{34}$$

where K is the complete elliptic integral of the first kind,

$$K(m) = \int_0^{\pi/2} (1 - m\sin^2\theta)^{-1/2}\,d\theta \tag{35}$$

and the parameter for both integrals is given by (31).

4. Extraordinary reflection and refraction

It is convenient to consider separately the cases when there is only an extraordinary wave transmitted into the material and when there is only an ordinary wave transmitted. The effect of an arbitrary incident wave can be found by superposing the fields found in these two cases.

Consider first the extraordinary wave as given by (25). On the surface of the crystal, at $z = 0$, the transverse fields inside are

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix}_{\text{ex}} = \frac{\bar{C}_{\text{ex}}}{[p_{\text{ex}}(0)]^{1/2}(1-\alpha\cos^2\Phi_0)^{1/2}} \begin{pmatrix} (1-\alpha)\cos\Phi_0 \\ \sin\Phi_0 \end{pmatrix} e^{i(lx-\omega t)} \tag{36}$$

$$\begin{pmatrix} B_x \\ B_y \end{pmatrix}_{\text{ex}} = \frac{\bar{C}_{\text{ex}}[p_{\text{ex}}(0)]^{1/2}}{\omega(1-\alpha\cos^2\Phi_0)^{1/2}} \begin{pmatrix} -\sin\Phi_0 \\ \cos\Phi_0 \end{pmatrix} e^{i(lx-\omega t)}.$$

Let the incident wave, in vacuum, be

$$\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}_{\text{inc}} = \frac{1}{[p_{\text{ex}}(0)]^{1/2}(1-\alpha\cos^2\Phi_0)^{1/2}} \begin{pmatrix} \bar{A}_x \\ \bar{A}_y \\ -(l/\bar{p})\bar{A}_x \end{pmatrix} e^{i(lx+\bar{p}z-\omega t)} \tag{37}$$

$$\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}_{\text{inc}} = \frac{\bar{p}/\omega}{[p_{\text{ex}}(0)]^{1/2}(1-\alpha\cos^2\Phi_0)^{1/2}} \begin{pmatrix} -\bar{A}_y \\ (\omega/c\bar{p})^2\bar{A}_x \\ (l/\bar{p})\bar{A}_y \end{pmatrix} e^{i(lx+\bar{p}z-\omega t)}$$

where $\bar{A}_x$ and $\bar{A}_y$ are constants to be determined and where $\bar{p}$ is a positive constant satisfying the dispersion equation

$$\omega^2 = c^2(l^2 + \bar{p}^2). \tag{38}$$

Also let the reflected wave be

$$\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}_{\text{refl}} = \frac{1}{[p_{\text{ex}}(0)]^{1/2}(1-\alpha\cos^2\Phi_0)^{1/2}} \begin{pmatrix} \bar{B}_x \\ \bar{B}_y \\ (l/\bar{p})\bar{B}_x \end{pmatrix} e^{i(lx-\bar{p}z-\omega t)}$$

$$\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}_{\text{refl}} = \frac{\bar{p}/\omega}{[p_{\text{ex}}(0)]^{1/2}(1-\alpha\cos^2\Phi_0)^{1/2}} \begin{pmatrix} \bar{B}_y \\ -(\omega/c\bar{p})^2\bar{B}_x \\ (l/\bar{p})\bar{B}_y \end{pmatrix} e^{i(lx-\bar{p}z-\omega t)}. \tag{39}$$

The components of the various wavevectors are illustrated in figure 2.

The boundary conditions are that the transverse components of $\boldsymbol{E}$ and $\boldsymbol{B}$ should be continuous. They give the equations

$$\begin{aligned} \bar{A}_x + \bar{B}_x &= \bar{C}_{\text{ex}}(1-\alpha)\cos\Phi_0 \\ \bar{A}_y + \bar{B}_y &= \bar{C}_{\text{ex}}\sin\Phi_0 \\ \bar{p}(-\bar{A}_y + \bar{B}_y) &= -\bar{C}_{\text{ex}}p_{\text{ex}}(0)\sin\Phi_0 \\ \bar{p}(\omega/c\bar{p})^2(\bar{A}_x - \bar{B}_x) &= \bar{C}_{\text{ex}}p_{\text{ex}}(0)\cos\Phi_0. \end{aligned} \tag{40}$$

These are easily solved, leading to the final results

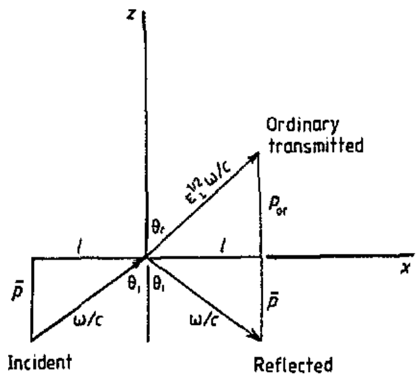

Figure 2. Wavevectors in the plane of reflection and refraction, for the incident beam, the reflected beam, and the ordinary transmitted beam.

$$\begin{aligned}\bar A_x &= \tfrac12[1-\alpha+\bar p p_{\rm ex}(0)/(\omega/c)^2]\bar C_{\rm ex}\cos\Phi_0\\ \bar A_y &= \tfrac12[1+p_{\rm ex}(0)/\bar p]\bar C_{\rm ex}\sin\Phi_0\\ \bar B_x &= \tfrac12[1-\alpha-\bar p p_{\rm ex}(0)/(\omega/c)^2]\bar C_{\rm ex}\cos\Phi_0\\ \bar B_y &= \tfrac12[1-p_{\rm ex}(0)/\bar p]\bar C_{\rm ex}\sin\Phi_0\end{aligned} \tag{41}$$

where

$$p_{\rm ex}(0) = (\omega/c)[(1-\alpha)\varepsilon_\parallel + (\varepsilon_\parallel - \varepsilon_\perp)\alpha\sin^2\Phi_0]^{1/2} \tag{42}$$

$$\bar p = (\omega/c)(1-\alpha\varepsilon_\perp)^{1/2}. \tag{43}$$

It is interesting that the $\bar A$ and $\bar B$ are real so the fields are all plane-polarized.

A question is how to find the angle of refraction, given the angle of incidence. For the incident wave the dispersion equation is (38), so the group velocity is c^2/ω times $(l, 0, \bar p)$. The angle of incidence θ_i is the angle between the group velocity and the z axis. For it

$$\sin^2\theta_i = l^2/(l^2+\bar p^2) = \varepsilon_\perp\alpha. \tag{44}$$

For the transmitted wave the group velocity of the coinciding wave is known, equation (34). Consequently, for the angle of refraction,

$$\tan\theta_r = V_x/V_z = (2/\pi)[E + (\varepsilon_\parallel/\alpha\varepsilon_\perp)(K-E)]/[(\varepsilon_\parallel/\alpha\varepsilon_\perp)-1]^{1/2}. \tag{45}$$

Given θ_i, one finds α from (44) and then θ_r from (45). For small angles, nearly normal incidence, this gives

$$\theta_r = [(\varepsilon_\parallel + \varepsilon_\perp)/2\varepsilon_\perp\varepsilon_\parallel^{1/2}]\theta_i. \tag{46}$$

There is a special set of conditions, at which there is no reflected wave. All the incident light is transmitted into the crystal. Consider $\sin\Phi_0 = 0$, so the director at the entry face is in the plane of the reflection and refraction. Then, from (41), it is seen that $\bar B_y$ is zero and that $\bar B_x$ is zero if the $(1-\alpha)$ and $\bar p p_{\rm ex}(0)/(\omega/c)^2$ terms cancel. This translates into the condition

$$\sin\theta_i = [(\varepsilon_\parallel\varepsilon_\perp - \varepsilon_\perp)/(\varepsilon_\parallel\varepsilon_\perp - 1)]^{1/2}. \tag{47}$$

In summary, if the incident wave is at this angle, plane-polarized parallel to the plane of refraction, and if the director is parallel to the plane of refraction on the crystal face, then there is no reflected wave.

5. Ordinary reflection and refraction

The same process applies in the case when there is only the ordinary wave transmitted into the material. Let this wave be given by (24). With the dispersion of (19), the group velocity is $c^2/\omega\varepsilon_\perp$ times $(l, 0, p_{or})$, so, for the angle of refraction,

$$\sin^2\theta_r = l^2/(l^2 + p_{or}^2) = \alpha. \tag{48}$$

Equations (44) and (48) together imply that Snell's law holds with $\varepsilon_\perp$ as the effective dielectric constant,

$$\sin\theta_i = \varepsilon_\perp^{1/2}\sin\theta_r. \tag{49}$$

Let the electric fields of the incident and reflected waves be

$$\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}_{\text{inc}} = \frac{1}{(1-\alpha\cos^2\Phi_0)^{1/2}}\begin{pmatrix} \bar{F}_x \\ \bar{F}_y \\ -(l/\bar{p})\bar{F}_x \end{pmatrix} e^{i(lx+\bar{p}z-\omega t)} \tag{50}$$

$$\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}_{\text{refl}} = \frac{1}{(1-\alpha\cos^2\Phi_0)^{1/2}}\begin{pmatrix} \bar{G}_x \\ \bar{G}_y \\ (l/\bar{p})\bar{G}_x \end{pmatrix} e^{i(lx-\bar{p}z-\omega t)}. \tag{51}$$

The results of matching the boundary conditions are

$$\begin{aligned} \bar{F}_x &= -\tfrac{1}{2}(1+\varepsilon_\perp\bar{p}/p_{or})\bar{C}_{or}\sin\Phi_0 \\ \bar{F}_y &= \tfrac{1}{2}(1+p_{or}/\bar{p})\bar{C}_{or}\cos\Phi_0 \\ \bar{G}_x &= -\tfrac{1}{2}(1-\varepsilon_\perp\bar{p}/p_{or})\bar{C}_{or}\sin\Phi_0 \\ \bar{G}_y &= \tfrac{1}{2}(1-p_{or}/\bar{p})\bar{C}_{or}\cos\Phi_0. \end{aligned} \tag{52}$$

Again there is a special situation in which there is no reflected wave. If $\cos\Phi_0$ is zero and the angle of incidence satisfies

$$\sin\theta_i = [\varepsilon_\perp/(1+\varepsilon_\perp)]^{1/2} \tag{53}$$

then both $\bar{G}_x$ and $\bar{G}_y$ are zero. Thus if the incident wave is at this special angle, plane-polarized parallel to the plane of refraction, and if the director is perpendicular to the plane of refraction on the crystal face, then there is no reflected wave.

6. Discussion

The system is linear, so a linear combination of solutions is also a solution. Which linear combination occurs depends on how the system is excited. For given frequency and angle of incidence, fixing ω, l and α, the general solution of the reflection–refraction problem is the sum of the ordinary and extraordinary solutions with arbitrary values of $\bar{C}_{\mathrm{or}}$ and $\bar{C}_{\mathrm{ex}}$. The incident radiation in this general case is

$$\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}_{\mathrm{inc}} = \frac{\bar{C}_{\mathrm{ex}}/2[p_{\mathrm{ex}}(0)]^{1/2}}{(1-\alpha\cos^2\Phi_0)^{1/2}} \begin{pmatrix} [1-\alpha+\bar{p}p_{\mathrm{ex}}(0)/(\omega/c)^2]\cos\Phi_0 \\ [1+p_{\mathrm{ex}}(0)/\bar{p}]\sin\Phi_0 \\ -(l/\bar{p})[1-\alpha+\bar{p}p_{\mathrm{ex}}(0)/(\omega/c)^2]\cos\Phi_0 \end{pmatrix} \mathrm{e}^{\mathrm{i}(lx+\bar{p}z-\omega t)}$$

$$+\frac{\bar{C}_{\mathrm{or}}/2}{(1-\alpha\cos^2\Phi_0)^{1/2}} \begin{pmatrix} -(1+\varepsilon_\perp\bar{p}/p_{\mathrm{or}})\sin\Phi_0 \\ (1+p_{\mathrm{or}}/\bar{p})\cos\Phi_0 \\ (l/\bar{p})(1+\varepsilon_\perp\bar{p}/p_{\mathrm{or}})\sin\Phi_0 \end{pmatrix} \mathrm{e}^{\mathrm{i}(lx+\bar{p}z-\omega t)}.$$

The first term here is found by substituting $\bar{A}_x$ and $\bar{A}_y$ from (41) into (37) and the second term by substituting $\bar{F}_x$ and $\bar{F}_y$ from (52) into (50). As an example, if one makes the special choice

$$\bar{C}_{\mathrm{ex}} = [p_{\mathrm{ex}}(0)]^{1/2}(1+\varepsilon_\perp\bar{p}/p_{\mathrm{or}})\sin\Phi_0$$
$$\bar{C}_{\mathrm{or}} = [1-\alpha+\bar{p}p_{\mathrm{ex}}(0)/(\omega/c)^2]\cos\Phi_0$$

then the incident beam has the form

$$\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}_{\mathrm{inc}} = (\mathrm{const})\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \mathrm{e}^{\mathrm{i}(lx+\bar{p}z-\omega t)}$$

so is plane-polarized in the y direction, normal to the plane of reflection and refraction. The reflected radiation is also plane-polarized; a formula for it would be complicated. The transmitted radiation has two components with different group velocities and angles of refraction, as given in (45) and (48). For any incident polarization of interest, one can determine appropriate values of $\bar{C}_{\mathrm{ex}}$ and $\bar{C}_{\mathrm{or}}$ and find the reflected and transmitted radiations.

The ideas developed in [8] about the applicability of the approximation used above will apply here similarly. The approximation requires that the local z wavelength $\lambda = 2\pi/p$ be slowly varying, $|\mathrm{d}\lambda/\mathrm{d}z| < 4\pi$, and that the free-space wavelength be small compared to the pitch of the director, $(c/\omega)|\mathrm{d}\Phi/\mathrm{d}z| < 1$.

For the ordinary wave the local z wavelength is constant, so it is slowly varying as required. For the extraordinary wave one finds

$$(1/4\pi)\,|\mathrm{d}\lambda_{\mathrm{ex}}/\mathrm{d}z| = k_1^3|q|\alpha\sin(2\Phi)/2p_{\mathrm{ex}}^3 \qquad (54)$$

which is of order $|q|/(\omega/c)$. Thus all conditions are met, and the approximation is valid, if only $|q|/(\omega/c)$ is small. This is the ratio of the free-space wavelength $2\pi c/\omega$ to the pitch $2\pi/|q|$ of the director.

This applicability is illustrated by the case of straight-through propagation, where the exact solution is known. As reviewed by de Gennes [10] that solution is

$$E_x = \tfrac{1}{2}(a e^{iqz} + b e^{-iqz})\, e^{i(pz-\omega t)}$$
$$E_y = -\tfrac{1}{2}i(a e^{iqz} - b e^{-iqz})\, e^{i(pz-\omega t)} \tag{55}$$

where the dispersion relation is

$$(p^2 - k_1^2 - k_2^2 + q^2)^2 - 4q^2p^2 - k_1^4 = 0. \tag{56}$$

This has the two solutions

$$p^2 = k_1^2 + k_2^2 + q^2 \pm (4q^2k_1^2 + 4q^2k_2^2 + k_1^4)^{1/2}. \tag{57}$$

The k are proportional to ω/c so for small $|q|/(\omega/c)$ this becomes

$$p^2 = (k_1^2 + k_2^2 \pm k_1^2)[1 + O(c^2q^2/\omega^2)]. \tag{58}$$

There is agreement with (18) and (17) which, when l and consequently α are zero, become

$$p^2 = 2k_1^2 + k_2^2 \qquad p^2 = k_2^2. \tag{59}$$

This approximation does not give information on the widths of the stop bands but it does give an estimate of their location. It is expected that the approximation will break down, and there will not be propagating solutions, if the phase change of a propagating wave in a periodicity distance $\pi/|q|$ were to be an integer, say $\bar{n}$, times π. For the extraordinary wave, as given by (25), the condition is

$$\int_0^{\pi/|q|} p_{ex}\, dz = \bar{n}\pi$$

or

$$(2\omega/|q|c)(\varepsilon_\parallel - \alpha\varepsilon_\perp)^{1/2} E(m) = \bar{n}\pi. \tag{60}$$

For the ordinary wave of (24) the condition is

$$p_{or}\pi/|q| = \bar{n}\pi. \tag{61}$$

One can express p_{or} as

$$p_{or} = (\omega/c)\varepsilon_\perp^{1/2} \cos\theta_r$$

so the condition is equivalently

$$2(\pi/|q|)\cos\theta_r = \bar{n}(2\pi\varepsilon_\perp^{-1/2} c/\omega) \tag{62}$$

which is the Bragg reflection rule for an effective wavelength $(2\pi\varepsilon_\perp^{-1/2} c/\omega)$.

7. Conclusions

In summary, a complete solution for the electromagnetic waves in the crystal, leading to concise new formulae for all the fields, has been found, in the limit when the free-space wavelength of the radiation is small compared to the pitch of the crystal. It is found that, in this limit, there are two types of solution: an ordinary wave with electric field always perpendicular to the director and an extraordinary wave with magnetic field always perpendicular to the director. The ordinary wave has a definite phase and group velocity; the extraordinary wave does not. However, there is a wave that coincides periodically with the extraordinary wave and can be used to keep track of the extraordinary effects. As a result of having this complete solution, one can analyse the reflection–refraction process in more detail than previously known:

(i) It is found that, in general, a narrow incident beam will give rise to two transmitted beams, ordinary and extraordinary, travelling separately in different directions. This is the same phenomenon as occurs with a homogeneous non-isotropic uniaxial crystal.

(ii) The ordinary and extraordinary angles of refraction are determined, equations (45), (46) and (49).

(iii) It is discovered that there are analogues of the Brewster's angle phenomenon, special conditions at which there is no reflected radiation, equations (47) and (53).

(iv) The amplitudes of the ordinary and extraordinary transmitted waves are found, for any direction and polarization of the wave incident on the crystal face.

As another aspect of the complete solution, formulae for the frequencies of the stop bands, at which there are no propagating solutions, are found, as a function of the angle of incidence, equations (60) and (62). Incident radiation at these angles and frequencies will be anomalously reflected from the crystal.

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